Due Fri

4.8 – Row Space, Column Space, and Null Space

Definition: A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ and an associated nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ are **corresponding linear systems**.

#28 Find a general solution of the system, and use that solution to find a general solution of the associated homogeneous system and a particular solution of the given system.

$A: 3\times 4$

$$\underbrace{\begin{bmatrix} 9 & -3 & 5 & 6 \\ 6 & -2 & 3 & 1 \\ 3 & -1 & 3 & 14 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix}}_{\vec{b}}$$

$\vec{x} \in \mathbb{R}^4$

$\vec{b} \in \mathbb{R}^3$

$$\left[\begin{array}{cccc|c} 9 & -3 & 5 & 6 & 4 \\ 6 & -2 & 3 & 1 & 5 \\ 3 & -1 & 3 & 14 & -8 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1/3 & 0 & -13/3 & 13/3 \\ 0 & 0 & 1 & 9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

If the system were homogeneous, we would have $x_1 = \frac{1}{3}x_2 + \frac{13}{3}x_4$ and $x_3 = -9x_4$

For $s = x_2$, $t = x_4$, we have

$$\vec{x} = \begin{bmatrix} \frac{1}{3}s + \frac{13}{3}t \\ s \\ -9t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} \frac{13}{3} \\ 0 \\ -9 \\ 1 \end{bmatrix} t$$

With the constants from $\vec{b}$, we get

$x_1 = \frac{1}{3}x_2 + \frac{13}{3}x_4 + \frac{13}{3}$, $x_3 = -9x_4 - 7$, so general solution to the system

$$\vec{x} = \underbrace{\begin{bmatrix} 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 13/3 \\ 0 \\ -9 \\ 1 \end{bmatrix} t}_{\text{General solution to the homogeneous system, } \vec{x}_h} + \underbrace{\begin{bmatrix} 13/3 \\ 0 \\ -7 \\ 0 \end{bmatrix}}_{\text{particular solution to the nonhomogeneous system, } \vec{x}_0} = \vec{x}_h + \vec{x}_0$$

Definition: The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is called the **general solution**. A constant vector $\mathbf{x}_0$ in the solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ is a **particular solution**.

solution space for the homogeneous system

Theorem 4.8.2 If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$. Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

Note that the solution space of $A\vec{x} = \vec{b}$ can be considered a translation of the solution space of $A\vec{x} = \vec{0}$ (null space) by $\vec{x}_0$, the particular solution vector.

Pf: Let $W = \text{span}(S) = \text{null}(A)$, $\vec{x}_0$ be a solution of $A\vec{x} = \vec{b}$, and $\vec{x} = \vec{x}_0 + W$, the set of vectors obtained by adding $\vec{x}_0$ to all vectors in W.

($\Rightarrow$) Let $\vec{x}$ be any solution of $A\vec{x} = \vec{b}$ and let $\vec{w} \in W$, so that $A\vec{w} = \vec{0}$.

For $\vec{w} = \vec{x} - \vec{x}_0$, we have

$$A\vec{w} = A(\vec{x} - \vec{x}_0) = A\vec{x} - A\vec{x}_0$$
$$= \vec{b} - \vec{b} = \vec{0}$$

so $\vec{x} = \vec{x}_0 + \vec{w} = \vec{x}_0 + \sum_{i=1}^{k} c_i \vec{v}_i$.

($\Leftarrow$) Suppose $\vec{x} \in \vec{x}_0 + W$. Then $\vec{x} = \vec{x}_0 + \vec{w}$ for some $\vec{w} \in W$.

$$A\vec{x} = A(\vec{x}_0 + \vec{w}) = A\vec{x}_0 + A\vec{w}$$
$$= \vec{b} + \vec{0} = \vec{b}$$

Thus, $\vec{x}$ is a solution to $A\vec{x} = \vec{b}$. ✓

The vectors in the general solution are linearly independent and so form a basis for the solution space of the homogeneous system. This is called the null space.

Definition: For an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ the vectors

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \mathbf{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ \vdots & \qquad\qquad \vdots \\ \mathbf{r}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

in R^n that are formed from the rows of A are called the **row vectors** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ \ldots, \ \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m that are formed from the columns of A are called the **column vectors** of A.

$T_A : R^n \to R^m$

Definition: If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is denoted by row (A) and is called the **row space** of A, and the subspace of R^m spanned by the column of vectors of A is denoted by col (A) and is called the **column space** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is denoted by null (A) and is called the **null space** of A. transf.

$A\vec{x} = \vec{0}$ $A\vec{x} = \vec{0}$ system of eqns. $T_A(\vec{x}) = \vec{0}$

matrix vector

null space   solution space   kernel

$T_A : \mathbb{R}^n \to \mathbb{R}^m$

Theorem 4.8.1 A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

$$\begin{bmatrix} 9 & -3 & 5 & 6 \\ 6 & -2 & 3 & 1 \\ 3 & -1 & 3 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} x_3 + \begin{bmatrix} 6 \\ 1 \\ 14 \end{bmatrix} x_4 = \begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix}$$

Theorem 4.8.3

a) Row equivalent matrices have the same row space.

b) Row equivalent matrices have the same null space.

If $A \xrightarrow[\text{ops}]{\text{(elem) row}} B$, then $\text{row}(A) = \text{row}(B)$

$A\vec{x} = \vec{0}$ $\quad A \to \text{rref}$ soln to homog system

Theorem 4.8.4 If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

Row reduce and look for pivots.

#9 Find bases for the null space and row space of A.

→ solution set to homog. system

a. $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

$$A \longrightarrow R$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 16x_3$

$x_2 = 19x_3$

$$\vec{x} = \begin{bmatrix} 16t \\ 19t \\ t \end{bmatrix} = \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} t$$

Basis for null(A) is $\left\{ \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \right\}$.

Basis for row(A) is $\{ [1 \ 0 \ -16], [0 \ 1 \ -19] \}$.

Pf Thm 4.8.3: (a) Matrices A & B are row equivalent if one can be obtained from the other by a series of elementary row operations. Since vector spaces are closed under addition and scalar multiplication, $\text{row}(B) \subseteq \text{row}(A)$ and $\text{row}(A) \subseteq \text{row}(B)$.

(b) Elementary row operations do not change the solution of a homogeneous system. Thus if A & B are row equivalent, then the solution to $A\vec{x} = \vec{0}$ is the same as that to $B\vec{x} = \vec{0}$ and hence A & B have the same null space. ✓

Note: In general, elementary row operations do not preserve a column space.

b. $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = \frac{1}{2} t_3$

x_2 is free

$\vec{x} = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s$

Basis for null(A): $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Basis for row(A): $\{ [2 \ 0 \ -1] \}$.

Thm 4.8.3 & 4.8.4 give us

$A \xrightarrow[\text{ops}]{\substack{\text{(elem)} \\ \text{row}}} R$ → basis for row(R) and row(A)

→ basis for col(R) (but not col(A)).

Theorem 4.8.5 If A and B are row equivalent matrices, then:
a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

Elem. row ops do not change the dimension of a column space.

$A \xrightarrow[\text{ops}]{\substack{\text{(elem)}\\ \text{row}}} R$

Use columns of R that contain leading 1s to identify columns of A that form a basis for $\mathrm{col}(A)$

$A \longrightarrow R$

Basis for $\mathrm{col}(A)$

Basis for $\mathrm{null}(A)$
Basis for $\mathrm{row}(A)$

#15 Find a basis for the subspace of R^4 that is spanned by the given vectors.
(1, 1, 0, 0), (0, 0, 1, 1), (−2, 0, 2, 2), (0, −3, 0, 3)

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The given set is the desired basis.

$$2\vec{v}_1 - \vec{v}_2 = (2,-2,10,4)+(2,-3,-1,0)=(4,-5,9,4)$$

#17 Find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors.
$\mathbf{v}_1 = (1, -1, 5, 2)$, $\mathbf{v}_2 = (-2, 3, 1, 0)$, $\mathbf{v}_3 = (4, -5, 9, 4)$, $\mathbf{v}_4 = (0, 4, 2, -3)$,
$\mathbf{v}_5 = (-7, 18, 2, -8)$

Row echelon form gives us leading 1s

$$\begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{array}$$
$$\begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 8 & 15 \\ 0 & 1 & -1 & 4 & 11 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{ccccc} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{array} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

basis is $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$.

$$2\vec{w}_1 - \vec{w}_2 = \vec{w}_3$$
$$-\vec{w}_1 + 3\vec{w}_2 + 2\vec{w}_4 = \vec{w}_5$$

$$\boxed{\begin{aligned} \vec{v}_3 &= 2\vec{v}_1 - v_2 \\ \vec{v}_5 &= -\vec{v}_1 + 3\vec{v}_2 + 2v_4 \end{aligned}}$$

The equations $2\vec{w}_1 - \vec{w}_2 = \vec{w}_3$
and $-\vec{w}_1 + 3\vec{w}_2 + 2\vec{w}_4 = \vec{w}_5$
are called dependency equations.

Row operations preserve dependency equations.

Definition: An equation that expresses a column vector of a matrix A that does not contain a pivot as a linear combination of column vectors that contain pivots is a **dependency equation**.

#18 Find a basis for the row space of A that ~~consists entirely of row~~ vectors of A.

Basis for $\mathrm{col}(A^T)$

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \qquad A \to A^T \longrightarrow R$$

~~basis for row(A)~~

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Corresponding rows of A are
$\{[1\ 4\ 5\ 2], [2\ 1\ 3\ 0]\}$.